Mel'nikov Analysis of Homoclinic Chaos in a Perturbed sine - Gordon Equation

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Abstract

We describe and characterize rigorously the chaotic behavior of the sine–Gordon equation. The existence of invariant manifolds and the persistence of homoclinic orbits for a perturbed sine–Gordon equation are established. We apply a geometric method based on Mel'nikov's analysis to derive conditions for the transversal intersection of invariant manifolds of a hyperbolic point of the perturbed Poincaré map.

1 Introduction

Homoclinic orbits have long been identified as a possible source of chaos in nonlinear finite–dimensional dynamical systems. For example, when an integrable 2-dimensional system, containing a smooth connection between the invariant manifolds of a saddle fixed point is perturbed, this connection is generally broken and the invariant manifolds intersect transversally. This yields isolated homoclinic orbits of the perturbed system, near which one expects to find shift map embeddings and chaos.

In the theory of finite–dimensional dynamical systems, a standard method to convert numerical experiments of chaotic dynamics into a rigourous mathematical description is to construct Smale horseshoes in a neighborhood of a homoclinic orbit [25] and then use these horseshoes to establish the existence of an invariant set on which the motion is topologically equivalent to a Bernoulli shift on a number of symbols. Such constructions begin with the existence, persistence and breaking of homoclinic orbits, which in turn are obtained through Mel'nikov analysis.

However, in the case of infinite dimensional dynamical systems the situation is more complicated [24]. As is well–known, realistic physical systems are usually modelled by

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nonlinear PDEs, whose chaotic behaviour is generally very difficult to study analytically. In the last 10 years, D.W.McLaughlin and collaborators have studied the periodic spectral transform of integrable soliton systems and constructed representations of global objects such as homoclinic orbits and whiskered tori, extending the Mel'nikov analysis to the NLS equation [15], [17]-[19], as well as a system of coupled NLS [22], [23]. Recently, G. Haller has been developing an alternative prespective, under certain conditions, he proved the existence of manifolds of multi-pulse Silnikov type orbits homoclinic to the critical torus of NLS equation. He has applied to finite dimensional discretizations of the perturbed NLS equation [8], [9] and to the PDE case [10], [11].

In this article we describe and characterize with mathematical precision the chaotic behavior of the perturbed sine–Gordon equation (SG)

$$\text{Case I}\quad u_{tt} - c^2 u_{xx} - \sin u = \varepsilon(g_1(u) - a u_t) \tag{1.1a}$$

$$\text{Case II}\quad u_{tt} - c^2 u_{xx} - \sin u = \varepsilon(g_2(t) - a u_t) \tag{1.1b}$$

with even periodic spatially boundary conditions, ε a small perturbation parameter, $0 < \varepsilon \ll 1$. The perturbation term $g_1(u)$ is a smooth nonlinear function satisfying $g_1(0) = 0, \partial_u g_1(0) = 0$ and g_2 is time-periodic with period T.

The unperturbed sine–Gordon equation

$$u_{tt} - c^2 u_{xx} - \sin u = 0 \tag{1.2}$$

is an infinite–dimensional Hamiltonian system [24]. We denote by $\mathbb{H}^1$ the Sobolev space of functions of x which are L–periodic, even and square integrable with square integrable first derivative on $[0, L)$

$$\mathbb{H}^1 = \left\{ \mathbf{u} = (u, u_t) \,:\, \mathbf{u}(-x) = \mathbf{u}(x) = \mathbf{u}(x+L), \quad \int_0^L |\partial_x \mathbf{u}|^2 \,\mathrm{d}x < \infty \right\} \equiv \mathcal{X} \tag{1.3}$$

The system of sine-Gordon models the phase difference between two superconducting layers in a Josephson junction, [21]. Equation (1.2) can also be thought of as a model for the continuum limit of a chain of coupled pendula, including damping and driving, with $u(x,t)$ the angle from the vertical down position of the pendulum at the point x along the chain.

On the phase space $\mathbb{H}^1$, the Hamiltonian of the equation (1.2) is given by

$$H : \mathbb{H}^1 \longrightarrow \mathbb{R}, \quad H(\mathbf{u}) = \int_0^L \left(\frac{1}{2}(u_t^2 + c^2 u_x^2) + (1 + \cos u) \right) \mathrm{d}x$$

with an infinite number of commuting constants of motion. The common level sets of these constants of motion are generically infinite dimensional tori of maximal dimension. These constants of the motion have linearly independent gradients, at certain critical tori, which become linearly dependent when the critical tori have dimension lower than maximal (but otherwise arbitrary). Critical tori can be either stable or unstable. In the unstable case, the constant level sets have a saddle structure in a neighborhood of the

critical torus. An unstable critical torus has an unstable manifold of infinite dimension. If, a phase point on this manifold approaches the critical torus as $t \longrightarrow \pm\infty$, such a phase point is said to lie on an orbit which is homoclinic to the critical torus.

Bishop *et al.* [1] have studied the sine–Gordon attractor numerically for $g_2(t) = \Gamma\cos\omega t$ and the parameter region $0.1 \leq \varepsilon\Gamma, \varepsilon a \leq 1$. In that region the so–called "breather" solutions (i.e localised in space and periodic in time) interact and qualitatively the dynamics appears to be well described by a four mode truncation [8], [14].

Ercolani *et al.,* [2] have explored the existence of homoclinic orbits for the unperturbed system. To this end, they used the fact that system (1.2) admits a Lax pair representation and derived explicit formulas for all homoclinic orbits in terms of Bäcklund transformations. In these formulas, the use of the imaginary double points of the spectrum of Lax operator is very important, since it describes the homoclinic orbits in infinite–dimensional space.

However, it is very important to study the behavior of the trajectories of the perturbed problem (1.1a), (1.1b). Especially, it is interesting to ask: under which perturbations do the homoclinic orbits persist and how does one establish their persistence? In the finite–dimensional system, the method which is most often used to answer such questions is the one established by Mel'nikov [20], [25].

In this paper, we extend and apply this geometric method for the perturbed system (1.1a), (1.1b), to establish the persistence of homoclinic orbits through the transversal intersection of invariant manifolds of singular fixed point. In section 2, we describe the symplectic structure and briefly outline the theory of the $\varepsilon = 0$ (integrable) case. In fact, in the integrable setting, the associated spectral problem has been used to classify all instabilities and to construct representations of their homoclinic orbits and we derive an analytic expression for the gradient of the Floquet discriminant. Section 3 is devoted to describe the invariant manifolds for the perturbed system (1.1a), (1.1b). We study the spectrum of the linearised system in the neighborhood of the uniform solution $u = (0, 0)$ lies on the circle S. The linearized operator admits two pair of real eigenvalues $\lambda_0^{\pm}, \lambda_1^{\pm}$ and infinite number of imaginary eigenvalues $\lambda_j^{\pm} = \pm\, \mathrm{i} a_j$, $j{\geq}2$, thus the uniform solution $(0, 0)$ of the sine-Gordon equation in the whole space becomes a singular point of type saddle-focus. We rewrite our system in a more convenient form and prove an invariant manifold theorem for the perturbed sine-Gordon equation (1.1a). Also, we construct the corresponding Poincaré map for the time-periodic perturbed system (1.1b) and analyze the dynamics on the perturbed manifolds.

We derive, in section 4, a formula for the distance between the codimension two center-stable and unstable manifold of the critical torus S of the perturbed system. We prove the existence of persistent homoclinic orbit to the saddle-focus point $O_\varepsilon = O + \mathrm{O}(\varepsilon) \in S$ for the system (1.1a) and for (1.1b) with $g_2(t) = \Gamma\cos\omega t$, this problem has been studied numerically by Bishop *et al.* [1]. The derivation of the distance between the manifolds is an extension of Mel'nikov methods in infinite dimensional space. We prove the convergence of the Mel'nikov integrals in Case I and II. The existence of simple zeros of the distance function implies a transversal intersection of those manifolds, as long as certain additional non–degeneracy conditions are satisfied. These simple zeros

correspond to orbits that are asymptotic to the saddle-focus point O_ε in backward time following the curve which lies in the unstable manifold and in forward time following the curves on the center-stable manifold.

2 Discussion of the Integrable System

Consider the nonlinear wave equation (1.2) on a segment of length L with even periodic boundary conditions in a symplectic space $\mathcal{X}$ (1.3) with scalar product $\langle\, .\,,\,.\,\rangle$, endowed by a weakly symplectic form Ω given by

$$\Omega(\mathbf{u},\mathbf{u}^1) = \int_0^L (u_t^1 u - u_t u^1)\,\mathrm{d}x, \quad \forall \mathbf{u},\mathbf{u}^1 \in \mathcal{X} \tag{2.1}$$

Let us define a domain $\mathcal{P} \subset \mathcal{X}$ and a function $f \in C^1(\mathcal{P})$ and let $\nabla_\alpha f \in \mathcal{X}$ be the gradient of f with respect to the inner product in $\mathcal{X}$

$$Df(\alpha)w := \frac{1}{L}\int_0^L \nabla_\alpha f(\alpha(x))w(x)\,\mathrm{d}x$$

Denoting by

$$H(\mathbf{u}) = \int_0^L \left(\frac{1}{2}(u_t^2 + c^2 u_x^2) + (1+\cos u)\right)\,\mathrm{d}x \in C^1(\mathcal{X})$$

the Hamiltonian vector field V_H with Hamiltonian H and the map $V_H : \mathcal{P} \longrightarrow \mathcal{X}$ such that $V_H = J\nabla H(\mathbf{u})$, the equation (1.2) takes the form

$$\mathbf{u}_t = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} \dfrac{\delta H}{\delta u} \\ \dfrac{\delta H}{\delta u_t} \end{pmatrix} = \begin{pmatrix} v \\ c^2 u_{xx} + \sin u \end{pmatrix} \tag{2.2}$$

Defining the Poisson bracket

$$\{\, F, G \,\} = \int_0^L \left(\frac{\delta F}{\delta u}\frac{\delta G}{\delta u_t} - \frac{\delta F}{\delta u_t}\frac{\delta G}{\delta u}\right)\,\mathrm{d}x \tag{2.3}$$

we note that the evolution of any functional F, under the SG flow, is governed by

$$\frac{\mathrm{d}F}{\mathrm{d}t} = \{\, F, H \,\} \tag{2.4}$$

Obviously, the Hamiltonian H is conserved by the SG flow. The SG equation admits an infinite family of conserved functionals in involution with respect to the Poisson bracket (2.3). This fact allows the SG to be solved with the inverse scattering transform [4].

Let us denote by $D(V_H) = \{\mathbf{u} \in \mathcal{P}, V_H(\mathbf{u}) \in \mathcal{X}\}$ the domain of V_H in $\mathcal{P}$. We give the following definition which is useful for the study of invariant manifolds, contained in section 3.

Definition 2.1 *Let $\mathcal{P}_1 \subset \mathcal{P}$ be such that for every $\mathbf{u}_0 \in \mathcal{P}_1$ there exists a unique solution $\mathbf{u}(t) = \Phi^t(\mathbf{u}_0)$, $t \in \mathcal{I}$ of (2.2) with initial condition $\mathbf{u}_0 = \mathbf{u}(0)$. The set of the mappings*

$$\Phi^t : \mathcal{P}_1 \longrightarrow \mathcal{P}, \quad \mathbf{u}_0 \longrightarrow \Phi^t(\mathbf{u}_0) \quad t \in \mathcal{I}$$

is called the flow of equation (2.2).

We are interested in studying the dynamics of the perturbed system (1.2) on a submanifold, $\tilde{\mathcal{X}}$, of codimension 2, in the infinite–dimensional phase space $\mathcal{X}$, given by two functionally independent equations

$$\tilde{\mathcal{X}} = \Big\{ \mathbf{u} \in \mathcal{X} \ : \ \mathbf{u}_x(x,t) = 0 \Big\} \tag{2.5}$$

and (1.2) becomes a pendulum system, for which we know its dynamics. Thus, starting with a 2–dimensional subspace Π of the codimension 2 manifold $\tilde{\mathcal{X}}$, we are in a position to follow the trajectories of (1.2) and study their behavior in the infinite dimensional space.

The phase space of (1.2) can be described in terms of the spectrum of the linear operator (for a detailed description see [2], [5]):

$$L^{(x)}(\mathbf{u},\zeta) := -\mathrm{i}c \begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix} \frac{\mathrm{d}}{\mathrm{d}x} + \frac{\mathrm{i}}{4}(cu_x + u_t) \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} - \frac{1}{16\zeta} \begin{pmatrix} e^{\mathrm{i}u} & 0 \\ 0 & e^{-\mathrm{i}u} \end{pmatrix} - \zeta I \tag{2.6}$$

The SG equation arises as the compatibility conditions of the following Lax pair of linear operators

$$L^{(x)}(\mathbf{u},\zeta)\phi = 0, \quad L^{(t)}(\mathbf{u},\zeta)\phi = 0 \tag{2.7}$$

where

$$L^{(t)}(\mathbf{u},\zeta) := -\mathrm{i} \begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix} \frac{\mathrm{d}}{\mathrm{d}t} + \frac{\mathrm{i}}{4}(cu_x + u_t) \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + \frac{1}{16\zeta} \begin{pmatrix} e^{\mathrm{i}u} & 0 \\ 0 & e^{-\mathrm{i}u} \end{pmatrix} - \zeta I \tag{2.8}$$

I is the identity matrix, $\mathbf{u} = (u(x,t), u_t(x,t))$ the potential and $\zeta \in \mathbb{C}$ denotes the spectral parameter.

The spectrum of $L^{(x)}$, denoted by

$$\sigma(L^{(x)}) := \{\zeta \in \mathbb{C} : L^{(x)}\phi = 0, |\phi| < \infty, \forall x\}, \tag{2.9}$$

characterizes the solution of the SG, the potential $\mathbf{u}$ also satisfies the SG and is of spatial period L. To achieve the spectral analysis of $L^{(x)}$, we use Floquet Theory as follows [5]:

Let us consider the fundamental matrix $M(x, x_0; \mathbf{u}, \zeta)$ of $L^{(x)}$ as:

$$L^{(x)}(\mathbf{u},\zeta)M = 0, \quad M(x_0, x_0; \mathbf{u}, \zeta) = I \tag{2.10}$$

and the Floquet discriminant

$$\Delta(\mathbf{u},\zeta) := \mathrm{trace} M(x_0 + L, x_0; \mathbf{u}, \zeta)$$

The spectrum of $L^{(x)}$ is given by the following condition

$$\sigma(L^{(x)}) := \Big\{ \zeta \in \mathbb{C} \,:\, \Delta(\mathbf{u},\zeta) \in \mathbb{R}, \quad |\Delta(\mathbf{u},\zeta)| \leq 2 \Big\} \tag{2.11}$$

where the discriminant Δ is known to be analytic in both its arguments and is invariant along solutions of the SG equation:

$$\frac{\mathrm{d}}{\mathrm{d}t}\Delta(\mathbf{u}(t),\zeta) = 0$$

Thus for each ζ Δ is a sine-Gordon constant of the motion. Moreover, by varying ζ, we generate all of the sine-Gordon constants. Thus, the level sets $\mathrm{M}_{\mathbf{u}}$ are defined through Δ:

$$\mathrm{M}_{\mathbf{u}} = \Big\{ \mathbf{v} \in \mathcal{X} \,:\, \Delta(\mathbf{u},\zeta) = \Delta(\mathbf{v},\zeta), \quad \forall\zeta \in \mathbb{C} \Big\}$$

We have the following elements of the $\sigma(L^{(x)})$ which determine the nonlinear mode content of solutions of sine–Gordon equations and the dynamical stability of these modes:

(i) simple periodic (antiperiodic) spectrum $\sigma^s_+(\sigma^s_-)$

$$\sigma^s_\pm = \Big\{ \zeta^s \in \mathbb{C} \,:\, \Delta(\mathbf{u},\zeta)\Big|_{\zeta^s=\zeta^s_\pm} = \pm 2, \quad \frac{\mathrm{d}\Delta}{\mathrm{d}\zeta}\Big|_{\zeta^s=\zeta^s_\pm} \neq 0 \Big\}$$

(ii) double points of the simple periodic (antiperiodic) spectrum

$$\sigma^d = \Big\{ \zeta^d \in \mathbb{C} \,:\, \Delta(\mathbf{u},\zeta)\Big|_{\zeta^d} = \pm 2, \quad \frac{\mathrm{d}\Delta}{\mathrm{d}\zeta}\Big|_{\zeta^d} = 0, \quad \frac{\mathrm{d}^2\Delta}{\mathrm{d}\zeta}\Big|_{\zeta^d} \neq 0 \Big\}$$

One studies the stability of a spatially indepedent solution $u(x,t) = \mathrm{c}(t)$ of the sine-Gordon equation, a solution which is periodic in t. As is well known [7], an analytical formula for this solution is

$$u(x,t) = \mathrm{c}(t) = 2\sin^{-1}(m\,\mathrm{sn}(t;m)) \tag{2.12}$$

where $\mathrm{sn}(t;m)$ is the Jacobi elliptic function, and its modulus m measures the amplitude of oscillation. Since $\mathrm{c}(t)$ is indepedent of x, the linearization can be solved by a Fourier transformation from $x \longrightarrow k = k_j = 2\pi j/L$,

$$\mathcal{U}(x,t) = \sum_{j\in\mathbb{Z}} \exp[\mathrm{i}k_j x]\,\hat{\mathcal{U}}(k_j,t) \tag{2.13}$$

This analysis is yields the following result: modes $\hat{\mathcal{U}}(k_j,t)$ are exponentially growing in t iff $k_j^2 \in (\,0, m^2)$. That is, the x-indepedent solution $\mathrm{c}(t)$ is unstable to long-wavelength perturbations. The number K of unstable modes increases with the period L. We list the unstable modes

$$2\pi/L, 2(2\pi/L), \ldots, K(2\pi/L) = [m]$$

where $[m]$ denotes the largest integer multiple of $2\pi/L$.

As an example let us compute the spectrum for the spatially and temporally uniform solution $\mathbf{u} = (0,0)$. In this case, the first system of (2.7) has constant coefficients and is readily solved. The Floquet discriminant is given by

$$\Delta(\mathbf{u},\zeta) = 2\cos\Big[(\zeta + \frac{1}{16\zeta})\frac{L}{c}\Big] \tag{2.14}$$

Making use of the definition above, the continuous spectrum is given by the entire real axis as well as the curve $|\zeta|^2 = \frac{1}{16}$ in the complex ζ–plane. The simple periodic spectrum is given by

$$\zeta = \frac{1}{2}\Big(\frac{jc\pi}{L} \pm \sqrt{(\frac{jc\pi}{L})^2 - \frac{1}{4}}\Big), \quad j \in \mathbb{Z} \tag{2.15}$$

Each of these points is a double point embedded in the continuous spectrum and becomes complex if

$$0 \le (\frac{2\pi jc}{L})^2 \le 1 \tag{2.16}$$

The condition (2.16) is exactly the same as the condition for linearized instability (see section 3 below) and the complex double point is given by

$$\zeta_d = \frac{1}{4}\exp[\mathrm{i}\beta] \tag{2.17}$$

with

$$\beta = \tan^{-1}\frac{L}{2j\pi c}\sqrt{1 - (\frac{2jc\pi}{L})^2}$$

The Floquet discriminant is an invariant of the SG equation. This means that $\Delta(\mathbf{u},\zeta)$ satisfies the following Poisson bracket conditions

$$\{\Delta(\mathbf{u},\zeta), \Delta(\mathbf{u},\zeta')\} = 0, \quad \forall \zeta,\zeta' \in \mathbb{C}, \quad \{\Delta(\mathbf{u},\zeta), H(\mathbf{u})\} = 0, \quad \forall \zeta \in \mathbb{C} \tag{2.18}$$

and the grad of Δ is given by (see Appendix)

$$\begin{aligned}
\frac{\delta\Delta}{\delta u}(\mathbf{u},\zeta) &= \frac{\mathrm{i}}{4}\mathrm{trace}\Big[M(L)D_x[M^{-1}(x)\mathcal{I}M(x)] + \frac{1}{4c\zeta}M(L)M^{-1}(x)\mathcal{E}M(x)\Big] \\
\frac{\delta\Delta}{\delta u_t}(\mathbf{u},\zeta) &= -\frac{\mathrm{i}}{4c}\mathrm{trace}\Big[M(L)M^{-1}(x)\mathcal{I}M(x)\Big]
\end{aligned} \tag{2.19}$$

$$\mathcal{I} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad \mathcal{E} = \begin{pmatrix} 0 & e^{-\mathrm{i}u} \\ e^{\mathrm{i}u} & 0 \end{pmatrix}$$

where $M(L) = M(L, x_0, \mathbf{u}, \zeta)$. Making use of the fact that $(0,0)$ satisfies the SG equation and the spectrum $\sigma(L^{(x)})$ is invariant in t, provides a countable number of constants of motion. Let $u(x,t)$ denote a solution of the SG equation which is periodic in x with

period L and $U(x,t)$ denote one of the Bäcklund transform of u as defined by (A.2) in the Appendix. Then their Floquet discriminant satisfies

$$\Delta(\zeta,U)=\Delta(\zeta,u),\quad \forall\zeta\in\mathbb{C}$$

that is U and u lie on the same isospectral set in $\mathcal{X}$. When $u\in\mathcal{X}_{NQ}$ (the N-phase quasiperiodic waves), one component of the level set $\mathrm{M}_{\mathbf{u}}$ is an N-torus $\mathbb{T}^N=\mathrm{M}_{\mathbf{u}}\times\mathcal{X}_{NQ}$. Let $u\in\mathbb{T}^N$ have a fully complex double point ζ_d, as an example we return to the case $\mathbf{u}=O(=(0,0))$ with $\zeta_d=\frac{1}{4}e^{\mathrm{i}\beta}$. Let $U\in\mathrm{M}_{\mathbf{u}}$ as defined by

$$U(x,t)=4\tan^{-1}\Big(\tan\beta\cos\left(xc\cos\beta\right)\operatorname{sech}\left(t\sin\beta\right)\Big)\tag{2.20}$$

Then [2]

i. $U\notin\mathbb{T}^N$

ii. U is homoclinic to $\mathbb{T}^N$, i.e there exists $U^{\pm}\in\mathbb{T}^N$ such that $\|U(x,t)-U^{\pm}(x,t)\|\longrightarrow 0$ as $t\to\pm\infty$.

The expression (2.20) is homoclinic to the torus,

$$U(x,t)=\mathrm{O}[\,\exp(\,-\sqrt{1-c^2}|t|\,)\,],\quad \text{as}\quad t\longrightarrow\pm\infty\tag{2.21}$$

The solution (2.20) is a homoclinic orbit of the integrable SG equation and is related to the famous soliton type solution known as the "breather" [2], in the sense that it may be called a "spatial breather", since it is localized in t and periodic in space.

3 An Invariant Manifold Theorem

In this section, we state and give the proof of an invariant manifold theorem for the perturbed sine-Gordon equation. There exists a two dimensional subspace $\tilde{\mathcal{X}}$

$$\tilde{\mathcal{X}}=\Big\{\,u\in\mathbb{H}^1\;:\;u_x=0\,\Big\}$$

such that the set $\Pi=\tilde{\mathcal{X}}\cap\mathbb{H}^1$ is invariant under the flow of the perturbed system. The manifold Π is real symplectic with a nondegenerate 2-form Ω_Π which arises from the restriction of the symplectic structure to Π. For $\varepsilon=0$, the system (1.2) restricted to Π becomes a 1 degree-of-freedom completely integrable Hamiltonian system, pendulum system, for which we know its dynamics. Furthermore, the uniform solution $O=(0,0)$ is a saddle type for the reduced pendulum system which lies on the torus S. On the whole space this uniform solution becomes a singular point of type saddle-focus. The breather solution $U(x,t)$ of the full system approaches this set when $t\to\pm\infty$. Starting with the space Π, we are in a position to follow the trajectories of (1.1a) and study the behavior of its solution in the infinite dimensional space.

3.1 Linearized Analysis

The linearized system of the perturbed sine-Gordon in the neighborhood of the solution $O = (0, 0) \in S$ takes the following form for the Case I;

$$\mathbf{r}_t = A\mathbf{r} + \mathrm{K}_I(\mathbf{r}) \tag{3.1}$$

where $\mathbf{r} = (\tilde{r}, \tilde{r}_t)$

$$A = \begin{pmatrix} 0 & 1 \\ c^2\partial_x^2 + 1 & 0 \end{pmatrix}, \quad \mathrm{K}_I(\mathbf{r}) = \begin{pmatrix} 0 \\ -\varepsilon a\tilde{r}_t - \tilde{G}_I(\tilde{r};\varepsilon) \end{pmatrix}$$

and the function $\tilde{G}_I$ is defined as follows:

$$\tilde{G}_I(\tilde{r}) = \varepsilon g_1 (O + \tilde{r}) + \sin\tilde{r} - \tilde{r} = \mathrm{O}(\tilde{r}^3) \tag{3.2}$$

for the Case II,

$$\mathbf{r}_t = A\mathbf{r} + \mathrm{K}_{II}(\mathbf{r}) \tag{3.3}$$

where

$$\mathrm{K}_{II}(\mathbf{r}) = \begin{pmatrix} 0 \\ -\varepsilon a\tilde{r}_t - \tilde{G}_{II}(\tilde{r};\varepsilon) \end{pmatrix}$$

and the function $\tilde{G}_{II}$ is defined as follows:

$$\tilde{G}_{II} = \varepsilon g_2 (t) + \sin\tilde{r} - \tilde{r} = \mathrm{O}(\tilde{r}^3) \tag{3.4}$$

We can also combine the two equations (3.1) in a single equation, (respectively for the Case II)

$$\tilde{r}_{tt} - c^2\tilde{r}_{xx} - \tilde{r} + \varepsilon a\tilde{r}_t = \tilde{G}_I(\tilde{r}) \tag{3.5}$$

Let $\varepsilon = 0$, (3.5) becomes

$$\tilde{r}_{tt} - c^2\tilde{r}_{xx} - \tilde{r} = 0, \quad c^2 \in \left(\frac{1}{4}, 1\right) \tag{3.6}$$

Substituting $\tilde{r} = \hat{r}(t)e^{\mathrm{i}k_j x}$ with $k_j = \frac{2\pi j}{L}$, and j an arbitrary integer we obtain from (3.6)

$$\hat{r}'' - (1 - c^2k_j^2)\hat{r} = 0$$

This shows that the j-th mode grows exponentially if $c^2(2\pi j/L)^2 < 1, j = 0, 1$. Using now the even spatial periodic boundary condition we see that our problem has two unstable directions, corresponding to the unstable directions of the saddle point $(0, 0)$: The zeroth mode in the plane Π $\alpha_0 = 1$, which is the most unstable in the sense that it grows at the fastest rate and $j = 1$, $\alpha_1 = \sqrt{1 - (2\pi c/L)^2}$ off the subspace Π. Hence, the saddle point $O = (0, 0)$ has a 2–dimensional unstable manifold. If $c^2(2\pi j/L)^2 > 1$, we find infinite number imaginary eigenvalues $\lambda_j = \pm\mathrm{i}\alpha_j = \pm\mathrm{i}\sqrt{c^2(2\pi j/L)^2 - 1}$, $j \geq 2$.

In order to study the local behavior of solutions of the nonlinear system (3.1) we need to analyze the spectrum of the operator

$$\mathcal{L}_\varepsilon = \begin{pmatrix} 0 & 1 \\ c^2\partial_x^2 + 1 & -\varepsilon a \end{pmatrix}$$

We consider the eigenvalue problem $\mathcal{L}_\varepsilon e = \lambda e$ for the eigen–pairs $\{e(x), \lambda\}$. Using Fourier expansions of $e(x)$ in $\exp(\mathrm{i}k_j x)$ leads to a quadratic expression for the eigenvalues λ:

$$\lambda^2 + \lambda\varepsilon a - (1 - c^2 k_j^2) = 0, \quad k_j = 2\pi j/L, \quad j = 0, 1, 2, \ldots$$

For $j = 0, 1$, we have:

$$\begin{aligned} e_{\mathrm{s,u}} &= (1, \mp\sigma_j)^\perp \cos k_j x \\ \lambda^\varepsilon_{\mathrm{s,u}} &= -\frac{\varepsilon a}{2} \pm \sqrt{(\frac{\varepsilon a}{2})^2 + \sigma_j^2}, \quad \sigma_j = \sqrt{1 - c^2 k_j^2} \end{aligned}$$

while for $j \geq 2$, the eigenvalues come in complex conjugate pairs.

3.2 Invariant Manifolds-Case I

From the above analysis, we conclude that the function u may be written as a linear combination

$$u = \tilde{r}_{\mathrm{u}}\mathbf{e}_{\mathrm{u}} + \tilde{r}_{\mathrm{s}}\mathbf{e}_{\mathrm{s}} + \tilde{r}_{\mathrm{c}}$$

and in terms of these variables the equation $\tilde{r}_t = \mathcal{L}_e \tilde{r}$ splits into

$$\begin{aligned} \tilde{r}_{\mathrm{u},t} &= \lambda^\varepsilon_{\mathrm{u}}\tilde{r}_{\mathrm{u}} \\ \tilde{r}_{\mathrm{s},t} &= -\lambda^\varepsilon_{\mathrm{s}}\tilde{r}_{\mathrm{s}} \\ \tilde{r}_{\mathrm{c},t} &= \mathcal{A}\tilde{r}_{\mathrm{c}} \end{aligned} \tag{3.7}$$

where the operator $\mathcal{A}$ corresponds to the infinite number of center directions. In a neighborhood of the fixed point $(0, 0)$ which lies on a torus S, the nonlinear equation (3.1) can be viewed as a perturbation of the linear equation (3.7). Under the flow of this linear equation and for $\varepsilon = 0$, S has a 2–dimensional stable and unstable manifold together with a codimension 4 center manifold. We focus our attention on the center manifold $E^{\mathrm{c}}(S)$ together with center-stable $E^{\mathrm{cs}}(S)$ and center-unstable $E^{\mathrm{cu}}(S)$ manifolds, on which the solutions have growth rates bounded by

(i) $\exp[\,\sigma_j t/n\,]$, for $t > 0$, and $\exp[\,-\sigma_j t/n\,]$, for $t < 0$, $j = 0, 1$

(ii) $\exp[\,\sigma_j |t|/n\,]$, for $\forall t$, $j \geq 2$, $\sigma_j = \mathrm{i}\sqrt{(ck_j)^2 - 1}$

respectively, while for the operator $\mathcal{A}$ we have

$$\|\exp[\,\mathcal{A}t\,]\| \leq nC\exp[\,\sigma_j |t|/n\,]$$

for some integer n and positive constant C.

Let us now concentrate on the existence of locally invariant manifolds of the perturbed SG equation. To this end, we introduce a localization function, [16], such that for all c_1, c_2 with $0 < c_1 < c_2$ and integer $p \geq 0$ there is a function $\varphi_\delta \in C^p$, $\varphi_\delta(y) = \varphi(y/\delta)$:

$$\varphi : \mathbb{R} \longrightarrow \mathbb{R}, \quad \varphi(y) = \begin{cases} 1 & \text{for} \quad |y| \leq c_1 \\ 0 & \text{for} \quad |y| \geq c_2 \end{cases}$$

with $\sup_y |\varphi^{(\imath)}(y)| \leq c$ for all $\imath \leq p$ and suitable c. Hence, equation (1.1a) is written as a perturbation of (3.7) with $u = (\tilde{r}_\mathrm{s}, \tilde{r}_\mathrm{u}, \tilde{r}_\mathrm{c})$

$$\begin{aligned} \tilde{r}_{\mathrm{u},t} &= \lambda^\varepsilon_\mathrm{u} \tilde{r}_\mathrm{u} + P_\mathrm{u}(\tilde{r}, \delta; \varepsilon) \\ \tilde{r}_{\mathrm{s},t} &= -\lambda^\varepsilon_\mathrm{s} \tilde{r}_\mathrm{s} + P_\mathrm{s}(\tilde{r}, \delta; \varepsilon) \\ \tilde{r}_{\mathrm{c},t} &= \mathcal{A} \tilde{r}_\mathrm{c} + P_\mathrm{c}(\tilde{r}, \delta; \varepsilon) \end{aligned} \tag{3.8}$$

where P_u, P_s, P_c are nonlinear functions. This localisation has the effect of keeping the flow unchanged in the neighborhood of torus S. We are now in position to prove the existence of an infinite dimensional, locally invariant manifold for equation (1.1a) in the vicinity of torus S, local invariance means that solutions can only leave the manifold through its boundary. The manifold we identify for equation (1.1a) is a codimension 4 center manifold for the torus S for $\varepsilon = 0$, and continuous to exist for nonzero values, even through S is usually destroyed by the perturbations. We also show that the manifold possesses locally invariant stable and unstable manifolds which codimension 2.

The following theorem concerns the existence of the locally invariant manifolds described above.

Theorem 3.1 *There exists a δ neighborhood of $S \subset \mathbb{H}^1$ and $\varepsilon_0(\delta) > 0$ such that $\forall\, \varepsilon \in [\,0, \varepsilon_0)$ equation (3.8) has locally invariant center-stable and unstable manifolds of codimension 2*

$$\begin{aligned} W^\mathrm{s}_\mathrm{loc}(\mathcal{M}_\varepsilon) &= \{\, \tilde{r} \in \mathbb{H}^1 \,:\, \tilde{r}_\mathrm{u} \,=\, h^\mathrm{u}\,(\,\tilde{r}_\mathrm{s}, \tilde{r}_\mathrm{c}; \varepsilon)\,\} \\ W^\mathrm{u}_\mathrm{loc}(\mathcal{M}_\varepsilon) &= \{\, \tilde{r} \in \mathbb{H}^1 \,:\, \tilde{r}_\mathrm{s} \,=\, h^\mathrm{s}\,(\,\tilde{r}_\mathrm{u}, \tilde{r}_\mathrm{c}; \varepsilon)\,\} \end{aligned} \tag{3.9}$$

with $\tilde{r}_\mathrm{u} = (\tilde{r}_{\mathrm{u},0}, \tilde{r}_{\mathrm{u},1}), \tilde{r}_\mathrm{s} = (\tilde{r}_{\mathrm{s},0}, \tilde{r}_{\mathrm{s},1})$ and a codimension 4 center manifold

$$\mathcal{M}_\varepsilon = \{\, \tilde{r} \in \mathbb{H}^1 \,:\, \tilde{r}_\mathrm{u} \,=\, h^\mathrm{cu}\,(\,\tilde{r}_\mathrm{c}; \varepsilon),\, \tilde{r}_\mathrm{s} \,=\, h^\mathrm{cs}\,(\,\tilde{r}_\mathrm{c}; \varepsilon)\,\} \tag{3.10}$$

where $\tilde{r}_\mathrm{c}$ is defined all of $\mathbb{R}$ and satisfies $\sup_{t\in\mathbb{R}} \|\tilde{r}_\mathrm{c}\|_{\mathbb{H}^1} \leq \frac{\delta'}{2}$, while δ' belongs to a neighborhood of δ.

Proof. We write (3.8) in the form of integral equations

$$\tilde{r}_\mathrm{u}(t) \;=\; \exp[\,\lambda^\varepsilon_\mathrm{u}\,(t - t_\mathrm{u})\,]\tilde{r}_\mathrm{u}(t_\mathrm{u}) + \int_{t_\mathrm{u}}^t \exp[\,\lambda^\varepsilon_\mathrm{u}\,(t - \xi)\,] P_\mathrm{u}(\tilde{r}(\xi), \delta, \varepsilon)\,\mathrm{d}\xi$$

$$
\begin{aligned}
\tilde r_{\rm s}(t) &= \exp[\,-\lambda^{\varepsilon}_{\rm s}\,(t-t_{\rm s})\,]\tilde r_{\rm s}(t_{\rm s}) + \int_{t_{\rm s}}^{t} \exp[\,-\lambda^{\varepsilon}_{\rm s}\,(t-\xi)\,]P_{\rm s}(\tilde r(\xi),\delta,\varepsilon)\,{\rm d}\xi \\
\tilde r_{\rm c}(t) &= \exp[\,\mathcal{A}t\,]\tilde r_{\rm c}(0) + \int_{0}^{t} \exp[\,\mathcal{A}\,(t-\xi)\,]P_{\rm c}(\tilde r(\xi),\delta,\varepsilon)\,{\rm d}\xi
\end{aligned} \tag{3.11}
$$

From the gap in the growth rates and the definition of the flow F^t_ε, we characterize the invariant manifolds $W^{\rm s}_{\rm loc}(\mathcal{M}_\varepsilon)$, $W^{\rm u}_{\rm loc}(\mathcal{M}_\varepsilon)$ by

$$
\begin{aligned}
W^{\rm s}_{\rm loc}(\mathcal{M}_\varepsilon) &= \Big\{\bar r \in \mathbb{H}^1 \,:\, \sup\nolimits_{t\geq 0}\{\,\exp[\,-\sigma_1 t/\kappa\,]\|\mathrm{F}^t_\varepsilon(\bar r;\varepsilon)\|_{\mathbb{H}^1}\} < \infty\Big\} \\
W^{\rm u}_{\rm loc}(\mathcal{M}_\varepsilon) &= \Big\{\bar r \in \mathbb{H}^1 \,:\, \sup\nolimits_{t\leq 0}\{\,\exp[\,\sigma_1 t/\kappa\,]\|\mathrm{F}^t_\varepsilon(\bar r;\varepsilon)\|_{\mathbb{H}^1}\} < \infty\Big\}
\end{aligned}
$$

Focusing our attention upon $W^{\rm cs}_{\rm loc}(\mathcal{M}_\varepsilon)$, for $\hat r$ in a sphere B of arbitrary radius ϱ, we introduce the norm

$$
\|\tilde r\|_\mu := \sup_{t\geq 0}\{\exp[\,-\sigma_1 t/\mu\,]\|\tilde r\|_{\mathbb{H}^1}\}, \quad \varrho \leq \|\tilde r\|_{\mathbb{H}^1}
$$

For $\tilde r \in W^{\rm cs}_{\rm loc}(\mathcal{M}_\varepsilon)$, we have $\exp[-\lambda^\varepsilon_{\rm u} t_{\rm u}]|\tilde r_{\rm u}(t_{\rm u})| \to 0$ as $t_{\rm u} \to \infty$. Thus, on the center–stable manifold the integral equations (3.11) can be written as:

$$
\begin{aligned}
\tilde r_{\rm u}(t) &= \int_{\infty}^{t} \exp[\,\lambda^{\varepsilon}_{\rm u}\,(t-\xi)\,]P_{\rm u}(\tilde r(\xi),\delta,\varepsilon)\,{\rm d}\xi \\
\tilde r_{\rm s}(t) &= \exp[\,-\lambda^{\varepsilon}_{\rm s} t\,]\tilde r_{\rm s}(t_{\rm s}) + \int_{0}^{t} \exp[\,-\lambda^{\varepsilon}_{\rm s}\,(t-\xi)\,]P_{\rm s}(\tilde r(\xi),\delta,\varepsilon)\,{\rm d}\xi \\
\tilde r_{\rm c}(t) &= \exp[\,\mathcal{A}t\,]\tilde r_{\rm c} + \int_{0}^{t} \exp[\,\mathcal{A}\,(t-\xi)\,]P_{\rm c}(\tilde r(\xi),\delta,\varepsilon)\,{\rm d}\xi
\end{aligned} \tag{3.12}
$$

To prove the existence of $W^{\rm s}_{\rm loc}(\mathcal{M}_\varepsilon)$, we use Newton's iterations. Let $\tilde r^0 = 0$ and

$$
\begin{aligned}
\tilde r^{\rm m+1}_{\rm u}(t) &= \int_{+\infty}^{t} \exp[\,\lambda^{\varepsilon}_{\rm u}(t-\xi)]\,P_{\rm u}(\tilde r^{\rm m}(\xi),\delta;\varepsilon)\,{\rm d}\xi \\
\tilde r^{\rm m+1}_{\rm s}(t) &= \exp[\,-\,\lambda^{\varepsilon}_{\rm s} t\,]\tilde r_{\rm s} + \int_{0}^{t} \exp[\,-\,\lambda^{\varepsilon}_{\rm s}(t-\xi)]\,P_{\rm s}(\tilde r^{\rm m}(\xi),\delta;\varepsilon)\,{\rm d}\xi \\
\tilde r^{\rm m+1}_{\rm c}(t) &= \exp[\,\mathcal{A}t\,]\tilde r_{\rm c} + \int_{0}^{t} \exp[\,\mathcal{A}(t-\xi)]\,P_{\rm c}(\tilde r^{\rm m}(\xi),\delta;\varepsilon)\,{\rm d}\xi
\end{aligned} \tag{3.13}
$$

We note that $P(\tilde r,\delta;\varepsilon)$ is a smooth function whose terms are linear with coefficient ε or nonlinear and localised in a δ-neighborhood of S. If we let P' be the derivative of P, we obtain the following estimate

$$
\|P(\tilde r,\delta;\varepsilon)\|_{\mathbb{H}^1} \leq \|P'\|\,\|\tilde r\|_{\mathbb{H}^1} + \varepsilon
$$

where $\|P'\|$ is the supremum of the magnitude of P' which is equal to $C(\delta+\varepsilon)$.

For if $\|\tilde r^{\rm m}\|_{\kappa_0} \leq C$, we obtain from (3.13)

$$
\|\tilde r^{\rm m+1}(t)\|_{\mathbb{H}^1} \;\leq\; \kappa_0 C \exp[\,\sigma_1 t/\kappa_0\,]\Big(\,\|\tilde r_{\rm s}\|_{\mathbb{H}^1} + \|\tilde r_{\rm c}\|_{\mathbb{H}^1} + \varepsilon\,\Big)
$$

$$
\begin{aligned}
&+ \int_t^\infty \exp[\,\frac{\sigma_1}{2}(t-\xi)\,]C(\delta+\varepsilon)\|\tilde{r}^{\mathrm{m}}(\xi)\|_{\mathbb{H}^1}\,\mathrm{d}\xi \\
&+ \int_0^t \kappa_0 C(\delta+\varepsilon)\exp[\,\frac{\sigma_1}{2\kappa_0}(t-\xi)\,]\|\tilde{r}^{\mathrm{m}}(\xi)\|_{\mathbb{H}^1}\,\mathrm{d}\xi
\end{aligned}
$$

By using the bound of $\tilde{r}^{\mathrm{m}}$ we obtain

$$
\|\tilde{r}^{\mathrm{m}+1}(t)\|_{\mathbb{H}^1} \le C\Big[\,\|\tilde{r}_{\mathrm{s}}\|_{\mathbb{H}^1} + \|\tilde{r}_{\mathrm{c}}\|_{\mathbb{H}^1} + \varepsilon\kappa_0^2(\delta+\varepsilon)\|\tilde{r}^{\mathrm{m}}\|_{\kappa_0}\Big]\exp[\,\sigma_1 t/\kappa_0\,]
$$

where the constant C is independent of κ_0, ε and δ. Fix $\delta = \frac{\alpha}{\kappa_0^2}$, where $\alpha = C/4$, we obtain for all $\varepsilon < C/4\kappa_0^2$:

$$
\|\tilde{r}^{\mathrm{m}+1}(t)\|_{\kappa_0} \le C(\rho) + \frac{1}{2}\|\tilde{r}^{\mathrm{m}}\|_{\kappa_0}
$$

Thus, we have proved that the sequence $\tilde{r}^{\mathrm{m}}$ is well defined and $\|\tilde{r}^{\mathrm{m}}\|_{\kappa_0} \le 2C(\rho)$. Since, the nonlinear term is smooth, we have a similar estimate for the difference

$$
\|\tilde{r}^{\mathrm{m}+1} - \tilde{r}^{\mathrm{m}}\|_{\kappa_0} \le \frac{1}{2}\|\tilde{r}^{\mathrm{m}} - \tilde{r}^{\mathrm{m}-1}\|_{\kappa_0} \tag{3.14}
$$

which implies that $\tilde{r}^{\mathrm{m}} \to \tilde{r}$ and

$$
\|\tilde{r}\|_{\kappa_0} \le 2C(\,\varepsilon + \|\tilde{r}_{\mathrm{c}}\|_{\mathbb{H}^1} + \|\tilde{r}_{\mathrm{s}}\|_{\mathbb{H}^1}\,)
$$

We note that all terms in (3.12) are smooth, which entails that $\{\tilde{r}^{\mathrm{m}}\}$ is differentiable.

We observe that the derivative $D\tilde{r}^{\mathrm{m}}$ satisfies:

$$
\begin{aligned}
\|D\tilde{r}^{\mathrm{m}}(t)\|_{\mathbb{H}^1} \;&<\; C\exp[\,\sigma_1 t/\kappa_0\,] + C\int_t^\infty \exp\Big[\frac{\sigma_1}{2}(t-\xi)\Big]\|P'D\tilde{r}^{\mathrm{m}}\|_{\mathbb{H}^1}\,\mathrm{d}\xi \\
&+\; C\int_0^t \exp\Big[\frac{\sigma_1}{2\kappa_0}(t-\xi)\Big]\|P'D\tilde{r}^{\mathrm{m}}\|_{\mathbb{H}^1}\,\mathrm{d}\xi
\end{aligned}
$$

By using the bounds on P', we obtain

$$
\|D\tilde{r}^{\mathrm{m}+1}\|_{\kappa_0} \le C + \frac{1}{2}\|D\tilde{r}^{\mathrm{m}}\|_{\kappa_0}
$$

which implies $\|D\tilde{r}^{\mathrm{m}}\|_{\kappa_0} \le 2C$.

Now, we are in position to estimate the difference between two terms in the sequence $\{\tilde{r}^{\mathrm{m}}\}$, $\delta\tilde{r}^{\mathrm{m}} = \tilde{r}^{\mathrm{m}} - \tilde{r}^{\mathrm{m}-1}$. We note by the mean value theorem

$$
\|P'(\tilde{r}^{\mathrm{m}}) - P'(\tilde{r}^{\mathrm{m}-1})\tilde{r}\|_{\mathbb{H}^1} \le C\|\delta\tilde{r}^{\mathrm{m}}\|_{\mathbb{H}^1}\|\tilde{r}\|_{\mathbb{H}^1}
$$

We have:

$$
\|D\delta\tilde{r}^{\mathrm{m}+1}(t)\|_{\mathbb{H}^1} \;\le\; C\int_t^\infty \exp\Big[\,\frac{\sigma_1}{2}(t-\xi)\,\Big]\Big\{\,\|P'(D\delta\tilde{r}^{\mathrm{m}})\|_{\mathbb{H}^1}
$$

$$+ \quad (\, \|D\tilde{r}^{\mathrm{m}}\|_{\mathbb{H}^1} + \|D\tilde{r}^{\mathrm{m}-1}\|_{\mathbb{H}^1}\,)\|\delta\tilde{r}^{\mathrm{m}}\|_{\mathbb{H}^1} \Big\}\,\mathrm{d}\xi$$

$$+ \quad C\int_0^t \exp\Big[\,\frac{\sigma_1}{2\kappa_0}(t-\xi)\,\Big]\Big\{\,\|P'(D\delta\tilde{r}^{\mathrm{m}})\|_{\mathbb{H}^1}$$

$$+ \quad (\, \|D\tilde{r}^{\mathrm{m}}\|_{\mathbb{H}^1} + \|D\tilde{r}^{\mathrm{m}-1}\|_{\mathbb{H}^1}\,)\|\delta\tilde{r}^{\mathrm{m}}\|_{\mathbb{H}^1} \Big\}\,\mathrm{d}\xi \tag{3.15}$$

and

$$\|\delta\tilde{r}^{\mathrm{m}}\|_{\mathbb{H}^1}\|D\tilde{r}^{\mathrm{m}}\|_{\mathbb{H}^1} \leq \exp\Big[\,\frac{2\sigma_1}{\kappa_0}t\,\Big]\|\delta\tilde{r}^{\mathrm{m}}\|_{\kappa_0}\|D\tilde{r}^{\mathrm{m}}\|_{\kappa_0}$$

Then, we obtain

$$\|D\tilde{r}^{\mathrm{m}+1} - D\tilde{r}^{\mathrm{m}}\|_{\kappa_0/2} \leq C\|\delta\tilde{r}^{\mathrm{m}}\|_{\kappa_0} + \frac{1}{2}\|D\tilde{r}^{\mathrm{m}} - D\tilde{r}^{\mathrm{m}-1}\|_{\kappa_0}$$

and sequence $\{\tilde{r}^{\mathrm{m}}\}$ converges in C^1 using the $\|\cdot\|_{\kappa_0/2}$-norm. We repeat this procedure to find bounds on $\{D^j\tilde{r}^{\mathrm{m}}\}$ in the $\|\cdot\|_{\kappa/j}$-norm with $\kappa/j > 2$.

We obtain $\tilde{r} \in C^{\mathrm{r}}$ for $\mathrm{r} \leq \left[\frac{\kappa_0}{2}\right] - 1$. Therefore, we define

$$\tilde{r}_{\mathrm{u}} = h^{\mathrm{u}}(\tilde{r}_{\mathrm{s}}, \tilde{r}_{\mathrm{c}}; \varepsilon) = \int_\infty^0 \exp[\,-\lambda_{\mathrm{u}}^{\varepsilon}\xi\,]P_{\mathrm{u}}(\tilde{r}(\xi), \delta; \varepsilon)\,\mathrm{d}\xi$$

which is a C^{r} functional with $\|D\tilde{r}_{\mathrm{u}}\| \leq 1/2$ and

$$W^{\mathrm{s}}_{\mathrm{loc}}(\mathcal{M}_\varepsilon) = \{\,\tilde{r} \in \mathbb{H}^1 \,:\, \tilde{r}_{\mathrm{u}} \,=\, h^{\mathrm{u}}\,(\,\tilde{r}_{\mathrm{s}}, \tilde{r}_{\mathrm{c}}; \varepsilon)\,\}$$

is a C^{r} manifold of codimension 2, with $\tilde{r}_{\mathrm{u}} = (\tilde{r}_{\mathrm{u},0}, \tilde{r}_{\mathrm{u},1})$. Exactly analogous results are obtained for the center-unstable manifold, $W^{\mathrm{u}}_{\mathrm{loc}}(\mathcal{M}_\varepsilon)$.

The center manifold $\mathcal{M}_\varepsilon$ is described by $W^{\mathrm{s}}_{\mathrm{loc}}(\mathcal{M}_\varepsilon) \cap W^{\mathrm{u}}_{\mathrm{loc}}(\mathcal{M}_\varepsilon)$, and equivalent we need to solve the following system

$$\tilde{r}_{\mathrm{u}} = h^{\mathrm{u}}(\,\tilde{r}_{\mathrm{s}}, \tilde{r}_{\mathrm{c}}; \varepsilon), \quad \tilde{r}_{\mathrm{s}} = h^{\mathrm{s}}(\,\tilde{r}_{\mathrm{u}}, \tilde{r}_{\mathrm{c}}; \varepsilon) \tag{3.16}$$

we note that $\|Dh^{\mathrm{u,s}}\| \leq 1/2$ and (3.14). From the implicit function theorem, we obtain that the system (3.16) has a unique C^{r} solution given by

$$\tilde{r}_{\mathrm{u}} = h^{\mathrm{cu}}_\varepsilon(\tilde{r}_{\mathrm{c}}; \varepsilon), \quad \tilde{r}_{\mathrm{s}} = h^{\mathrm{cs}}_\varepsilon(\tilde{r}_{\mathrm{c}}; \varepsilon)$$

□

3.3 Invariant Manifolds-Case II

The time-periodic perturbed sine-Gordon equation (1.1b) can be rewritten as:

$$\mathbf{u}_t = \mathrm{A}\mathbf{u} + \mathrm{B}(\mathbf{u}) + \varepsilon\Big(\mathrm{C}\mathbf{u} + g_2(t)\Big) \tag{3.17}$$

with

$$\mathrm{A} := \begin{pmatrix} 0 & 1 \\ c^2\partial_x^2 & 0 \end{pmatrix}, \quad \mathrm{B}(u) := \begin{pmatrix} 0 \\ \sin u \end{pmatrix}, \quad \mathrm{C} := \begin{pmatrix} 0 & 0 \\ 0 & -a \end{pmatrix}$$

We assume that $g_2 : \mathbb{H}^1 \times \mathbb{S}^1 \to \mathbb{H}^1$ is C^∞ T-periodic in time where $\mathbb{S}^1 = \mathbb{R}/T$. The associated extended autonomous system in a cylinder in the space $\mathbb{H}^1 \times \mathbb{S}^1$

$$\begin{array}{rcl} \dot{u} &=& \mathrm{A}u + \mathrm{B}(u) + \varepsilon(\mathrm{C}u + g_2(\theta)) \\ \dot{\theta} &=& 1 \end{array} \tag{3.18}$$

has a smooth flow F_ε^t. The geometry of the unperturbed system (3.17) is the same with the case I.

Suppose that linearized system of (3.17) with respect to the singular point $p_0 = O$

$$\mathbf{r}_t = \mathcal{L}_\varepsilon \mathbf{r} + \varepsilon g_2(t) \tag{3.19}$$

with

$$\mathcal{L}_\varepsilon := \begin{pmatrix} 0 & 1 \\ c^2\partial_x^2 + 1 & -\varepsilon a \end{pmatrix}$$

has a T-periodic function $\tilde{r}(t,\varepsilon)$, such that $\tilde{r}(t,\varepsilon) = \mathrm{O}(\varepsilon)$.

For $\varepsilon > 0$, the operator $\mathrm{e}^{T\mathcal{L}_\varepsilon}$ has a spectrum consisting in two pair of real eigenvalues for $j = 0, 1$ and infinite number of complex eigenvalues for $j \geq 2$, $\mathrm{spec}(\mathrm{e}^{T\mathcal{L}_\varepsilon})_{j\geq 2}$, in such a way:

$$\mu_1\varepsilon \leq \mathrm{dist}\Big(\mathrm{spec}(\mathrm{e}^{T\mathcal{L}_\varepsilon})_{j\geq 2}, |z| = 1\Big) \leq \mu_2\varepsilon \tag{3.20}$$

with $\mu_1, \mu_2 > 0$.

We denote by $\mathrm{P}^\varepsilon : \mathbb{H}^1 \to \mathbb{H}^1$ the Poincaré map for the flow F_ε^t such that $\mathrm{P}^0(p_0) = p_0$ and p_ε the fixed points of P^ε correspond to periodic orbits of F_ε^t, near to $p_0 = O : p_\varepsilon = p_0 + \mathrm{O}(\varepsilon)$.

Indeed, by (3.19) we have:

$$\tilde{r}(t,\varepsilon) = \mathrm{e}^{t\mathcal{L}_\varepsilon}\tilde{r}(0,\varepsilon) + \int_0^t \varepsilon \mathrm{e}^{(t-\xi)\mathcal{L}_\varepsilon} g_2(\xi)\,\mathrm{d}\xi \tag{3.21}$$

and $\tilde{r}(T,\varepsilon) = \tilde{r}(0,\varepsilon)$, $\|\tilde{r}(t,\varepsilon)\| \leq \mu_3\varepsilon$.

We seek a solution $u(t,\varepsilon)$ of the system (1.1b) such that:

$$u(t,\varepsilon) = \mathrm{e}^{t\mathcal{L}_\varepsilon}u(0,\varepsilon) + \int_0^t \varepsilon \mathrm{e}^{(t-\xi)\mathcal{L}_\varepsilon} g_2(\xi)\,\mathrm{d}\xi + \int_0^t \varepsilon \mathrm{e}^{(t-\xi)\mathcal{L}_\varepsilon}(\mathrm{B}(u(\xi,\varepsilon)) - u(\xi,\varepsilon))\,\mathrm{d}\xi \tag{3.22}$$

From (3.21) and (3.22) we obtain:

$$u(t,\varepsilon) - \tilde{r}(t,\varepsilon) = \mathrm{e}^{t\mathcal{L}_\varepsilon}(u(0,\varepsilon) - \tilde{r}(0,\varepsilon)) + \int_0^t \varepsilon \mathrm{e}^{(t-\xi)\mathcal{L}_\varepsilon}(\mathrm{B}(u(\xi,\varepsilon)) - u(\xi,\varepsilon))\,\mathrm{d}\xi \tag{3.23}$$

Thus,

$$\|u(t,\varepsilon) - \tilde{r}(t,\varepsilon)\|_{\mathbb{H}^1} \leq \mu_4\varepsilon + \mu_5 \int_0^t \|u(\xi,\varepsilon)\|_{\mathbb{H}^1}\,\mathrm{d}\xi$$

using $\|\tilde r(t,\varepsilon)\|_{\mathbb{H}^1}\le\mu_3\varepsilon$. By Gronwall's estimate we obtain $\|u(t,\varepsilon)\|_{\mathbb{H}^1}\le\mu_6\varepsilon$.

We consider now the set B_ε as

$$B_\varepsilon=\Big\{\,\tilde r(t,\varepsilon):\|\tilde r(t,\varepsilon)-\tilde r(0,\varepsilon)\|_{\mathbb{H}^1}<\varepsilon\Big\}$$

and the map

$$\begin{array}{rcl}\mathrm{P}^\varepsilon\,:\,B_\varepsilon & \to & \mathbb{H}^1\\ u(0,\varepsilon) & \to & u(T,\varepsilon)\end{array}$$

We seek a fixed point of P^ε. From (3.23) $u(0,\varepsilon)$ is a fixed point of P^ε if and only if it is a fixed point of the following map

$$\mathbb{G}_\varepsilon\,:\,B_\varepsilon\to\mathbb{H}^1$$

with

$$\mathbb{G}_\varepsilon(u(0,\varepsilon))=\tilde r(0,\varepsilon)+\mathrm{L}_\varepsilon^{-1}\int_0^T \mathrm{e}^{(T\xi)\mathcal{L}_\varepsilon}\Big(\mathrm{B}(u(\xi,\varepsilon)-u(\xi,\varepsilon))\Big)\,\mathrm{d}\xi \tag{3.24}$$

where $\mathrm{L}_\varepsilon=(\mathrm{Id}-\mathrm{e}^{T\mathcal{L}_\varepsilon})$ is invertible operator and the condition

$$\|\mathrm{L}_\varepsilon^{-1}\|_{\mathbb{H}^1}\le\mu_7\varepsilon^{-1} \tag{3.25}$$

is inferred from (3.20). The conditions (3.24) and (3.25) entail

$$\|\mathbb{G}_\varepsilon(u(0,\varepsilon))-\tilde r(0,\varepsilon)\|_{\mathbb{H}^1}\le\varepsilon$$

which implies that for $\varepsilon\ll 1$ $\mathbb{G}_\varepsilon$ maps B_ε to itself. Also, $\mathbb{G}_\varepsilon$ is a contraction since, for ε small enough

$$\begin{aligned}\|D\mathbb{G}_\varepsilon(u(0,\varepsilon))\|_{\mathbb{H}^1} &= \Big\|\,\mathrm{L}_\varepsilon^{-1}\int_0^T \mathrm{e}^{(T-\xi)\mathcal{L}_\varepsilon}\,D(\mathrm{B}(u(\xi,\varepsilon)-u(\xi,\varepsilon)))\circ D\mathrm{F}^\varepsilon_\xi(u(0,\varepsilon))\,\mathrm{d}\xi\,\Big\|_{\mathbb{H}^1}\\ &\le \mu_8\varepsilon<1\end{aligned}$$

Thus, $\mathbb{G}_\varepsilon$ has a unique fixed point on B_ε. There is an analogue result for the Poinacré map $\mathrm{P}^\varepsilon_{t_0}$ with the section $\mathbb{H}^1\times\{t_0\}$ in $\mathbb{H}^1\times\mathbb{S}^1$.

In the following Lemma, we describe the geometry of the invariant manifolds for the non-autonomous perturbed sine-Gordon equation (1.1b). Its proof based on the invariant manifolds theory [12] and the smoothness of the flow operator F^t_ε.

Lemma 3.1 *Let $p_\varepsilon(t_0)$ denote the unique fixed point of $\mathrm{P}^\varepsilon_{t_0}$. Corresponding to the spectrum of $\mathrm{e}^{t\mathcal{L}_\varepsilon}$, there are unique invariant manifolds $W^{\mathrm{cs}}(p_\varepsilon(t_0))$ (center-stable) and $W^{\mathrm{u}}(p_\varepsilon(t_0))$ (unstable manifold) for the map $\mathrm{P}^\varepsilon_{t_0}$ such that they are invariant under the $\mathrm{P}^\varepsilon_{t_0}$, tangent to the eigenspaces of $\mathrm{e}^{t\mathcal{L}_\varepsilon}$ respectively at $p_\varepsilon(t_0)$ and C^r-close as $\varepsilon\to 0$ to the homoclinic orbit $U(x,t)$ of the sine-Gordon equation (1.2) for $t_{\mathrm{s}},t_{\mathrm{u}}$ such that $t_{\mathrm{s}}<t<\infty,-\infty<t<t_{\mathrm{u}}$, respectively. We denote by $q_\varepsilon(t)=(p_\varepsilon(t),t),t\in(0,T)$ the periodic orbit of the extended system (3.18) with $q_\varepsilon(0)=(p_\varepsilon,0)$. The invariant manifolds for the periodic orbit $q_\varepsilon(t)$ are denoted $W^{\mathrm{cs}}(q_\varepsilon(t)),W^{\mathrm{u}}(q_\varepsilon(t))$ and*

$$W^j(p_\varepsilon(t_0))=W^j(q_\varepsilon(t))\cap(\mathbb{H}^1\times\{t_0\}),\quad j=\mathrm{cs},\mathrm{u}$$

4 Mel'nikov Analysis

In this section, we shall combine the invariant manifold theory of the previous section with explicit global information from the unperturbed sine–Gordon equation to establish the existence of persistent homoclinic orbit to the singular point O_ε.

4.1 Geometry of Mel'nikov Method-Case I

We recall that the unstable manifold $W^{\mathrm{u}}(O_\varepsilon) \subset W^{\mathrm{u}}(\mathcal{M}_\varepsilon)$ is two-dimensional with one direction in the plane of spatial-independent solutions Π and the other direction off the plane in the function space $\mathbb{H}^1$. We consider an orbit in $W^{\mathrm{u}}(O_\varepsilon)$ that leaves O_ε and takes off away from the plane, then defines an orbit which lies on the codimension 2 center-unstable manifold. We will construct a distance function $\mathcal{D}$ between the stable and unstable manifolds, whose zeros correspond to orbits that do not lie on the plane Π and are asymptotic to the saddle-focus point O_ε in backward and forward time.

Let us rewrite equation (1.1a) in the vector form

$$\mathbf{u}_t \;=\; J\,\nabla_{\mathbf{u}} H(\mathbf{u}) \;+\; \varepsilon \mathrm{G}_I(\,\mathbf{u}, \mu\,) \tag{4.1}$$

where $\mathrm{G}_I(\,\mathbf{u}, \mu\,) \;=\; (0, b g_1(u) - a u_t\,))^{\perp}$ and $\mu = (a, b, c)$

For $\varepsilon \leq \varepsilon_0$, (3.8) is related to the original system (1.1a) within some fixed open set

$$Y = \Big\{\, (\tilde{r}_{\mathrm{s}}, \tilde{r}_{\mathrm{u}}, \tilde{r}_{\mathrm{c}}) \in \mathbb{H}^1 \,:\, |\tilde{r}_{\mathrm{s}}| < C_{\mathrm{s}},\, |\tilde{r}_{\mathrm{u}}| < C_{\mathrm{u}},\, \|\tilde{r}_{\mathrm{c}}\|_{\mathbb{H}^1} < C_{\mathrm{c}} \Big\}$$

where $C_{\mathrm{s}}, C_{\mathrm{u}}, C_{\mathrm{c}}$ are fixed positive constants. We are interested in the dynamics of the solution $u_\varepsilon(t) = (\tilde{r}_{\mathrm{s}}, \tilde{r}_{\mathrm{u}}, \tilde{r}_{\mathrm{c}})$ of (3.8) in forward and backward time. We define a neighborhood $\mathcal{U}$ of $\mathcal{M}_\varepsilon$

$$\mathcal{U} = \{\, (\tilde{r}_{\mathrm{s}}, \tilde{r}_{\mathrm{u}}, \tilde{r}_{\mathrm{c}}) \in Y : |\tilde{r}_{\mathrm{s}}| < \delta,\, |\tilde{r}_{\mathrm{u}}| < \delta,\, \|\tilde{r}_{\mathrm{c}}\|_{\mathbb{H}^1} < \delta \leq C_{\tilde{r}_{\mathrm{c}}} \}$$

and its boundary $\partial\mathcal{U}$ consists of two parts $\partial\mathcal{U}_{\mathrm{s}}$ stable and $\partial\mathcal{U}_{\mathrm{u}}$ unstable as follows

$$\begin{aligned} \partial\mathcal{U}_{\mathrm{s}} &= \{\, (\tilde{r}_{\mathrm{s}}, \tilde{r}_{\mathrm{u}}, \tilde{r}_{\mathrm{c}}) \in \mathcal{U} \,:\, |\tilde{r}_{\mathrm{s}}| = \delta \} \\ \partial\mathcal{U}_{\mathrm{u}} &= \{\, (\tilde{r}_{\mathrm{s}}, \tilde{r}_{\mathrm{u}}, \tilde{r}_{\mathrm{c}}) \in \mathcal{U} \,:\, |\tilde{r}_{\mathrm{u}}| = \delta\, \} \end{aligned}$$

where the solution curves enter through $\partial\mathcal{U}_{\mathrm{s}}$ and they exit through $\partial\mathcal{U}_{\mathrm{u}}$. As was said before, the breather solution $U(x, t)$ is also a solution of the SG and is homoclinic to the point $O = (0, 0) \in \Pi$ which has two unstable directions, (cf. section 3). The unperturbed system possesses a homoclinic orbit u which evolves along the unstable direction from the plane Π, intersects the unstable boundary $\partial\mathcal{U}_{\mathrm{u}}$ at the point u_0^{u} and defines a unique orbit $(\,u^{\mathrm{u}}(t), t)$ which lies on the codimension two center-unstable manifold. After a finite time, this homoclinic orbit u intersects the stable boundary $\partial\mathcal{U}_{\mathrm{s}}$ at the point u_0^{s} and comes back to the plane $\Pi \subset S$ and is asymptotic to the singular point O, following a unique curve $(u^{\mathrm{s}}(t), t)$ which lies on the center-stable manifold. Thus, for

$\varepsilon \neq 0$, there exists a perturbed orbit u_ε^u which evolves from the perturbed saddle point $O_\varepsilon = O + \mathrm{O}(\varepsilon) \in \Pi$ and intersects the unstable boundary at the point $u_\varepsilon^{\mathrm{u}}$.

Now, choose a point u_h on the homoclinic orbit for the unperturbed system. Let Σ be a codimension 2 hyperplane at u_h which is transversal to the homoclinic orbit u and which contains the vector $\vec{\nu} = (\frac{\delta\Delta}{\delta u}, \frac{\delta\Delta}{\delta u_t})$. $W^{\mathrm{s}}(\mathcal{M}_0) \cap \Sigma$ has a codimension 1 in Σ and $\vec{\nu}$ is transversal to $W^{\mathrm{s}}(\mathcal{M}_0) \cap \Sigma$. Let u_{s} be the intersection of the line through $u_{\mathrm{u}} \in W^{\mathrm{u}}(O_\varepsilon) \subset W^{\mathrm{u}}(\mathcal{M}_\varepsilon)$ and the manifolds $W^{\mathrm{s}}(\mathcal{M}_\varepsilon)$ along to direction $\vec{\nu}$. We may thus define the inner product

$$\mathcal{D} = \langle \nabla\Delta, u_{\mathrm{u}} - u_{\mathrm{s}} \rangle, \tag{4.2}$$

as a measure of the distance between u_{u} and u_{s}.

To calculate $\mathcal{D}$, we parametrise the orbits as follows,

$$\text{for} \quad t \leq 0, \quad u = u_h(t + t_*), \quad u^{\mathrm{u}} = u_\varepsilon^{\mathrm{u}}(t + t_*)$$

$$\text{for} \quad t \geq 0, \quad u = u_h(t + t_*), \quad u^{\mathrm{s}} = u_\varepsilon^{\mathrm{s}}(t + t_*)$$

where $u^{\mathrm{s}}, u^{\mathrm{u}}$ be the solutions of the localized equations (3.8). Since u_h remains in a δ-neighborhood of S for $t \geq t_*$ u is a solution of the localized system (3.8), we recall that the invariant manifold theorem holds in a δ-neighborhood of $S \subset \mathbb{H}^1$. For $t \leq -t_*$ both orbits u, u^{u} remain in a δ-neighborhood of S. We apply the Gronwall's estimate to system (3.8) and from $\|u^{\mathrm{u}} - u_h\|_{\mathbb{H}^1} \leq C_1\varepsilon$, for $t < 0$ we have,

$$\|u^{\mathrm{u}} - u_h\|_{\mathbb{H}^1} \leq \varepsilon C_2 \mathrm{e}^{-\sigma t} \tag{4.3}$$

Similar, for $t \geq 0$

$$\|u^{\mathrm{s}} - u_h\|_{\mathbb{H}^1} \leq \varepsilon C_2 \mathrm{e}^{\sigma t} \tag{4.4}$$

The above analysis allows to decompose (4.2) as follows;

$$\mathcal{D} = \langle \nabla\Delta(u_h), u_{\mathrm{u}} - u_h \rangle - \langle \nabla\Delta(u_h), u_{\mathrm{s}} - u_h \rangle.$$

Let

$$\mathcal{D}^{\mathrm{s}} = \langle \nabla\Delta, u_\varepsilon^{\mathrm{s}}(t) - u(t) \rangle, \quad t \geq 0$$

$$\mathcal{D}^{\mathrm{u}} = \langle \nabla\Delta, u_\varepsilon^{\mathrm{u}}(t) - u(t) \rangle, \quad t \leq 0$$

Then

$$\mathcal{D} = \mathcal{D}^{\mathrm{u}}(0) - \mathcal{D}^{\mathrm{s}}(0)$$

Proposition 4.1 *The distance function (4.2) is given by*

$$\mathcal{D} = \mathcal{D}^{\mathrm{u}}(0) - \mathcal{D}^{\mathrm{s}}(0) = \varepsilon\, M_I + \mathrm{O}(\varepsilon^2) \tag{4.5}$$

where

$$M_I = \int_{-\infty}^{\infty} \langle \nabla\Delta(u(t)), \mathrm{G}_I(u(t)) \rangle \, \mathrm{d}t, \tag{4.6}$$

denote the Mel'nikov integral with $\langle a, b \rangle = \int_0^L a(x)b(x)\, \mathrm{d}x$.

Proof. We start the proof by asserting from (4.1) that $u = u(t)$, $u^{\mathrm{u}}_\varepsilon = u^{\mathrm{u}}_\varepsilon(t)$, $u^{\mathrm{s}}_\varepsilon = u^{\mathrm{s}}_\varepsilon(t)$ solve the following system of equations

$$\begin{aligned} \partial_t u &= J\,\nabla H(u) \\ \partial_t u^{\mathrm{u}}_\varepsilon &= J\,\nabla H(u^{\mathrm{u}}_\varepsilon) + \varepsilon \mathrm{G}_I(u^{\mathrm{u}}_\varepsilon, \mu) \\ \partial_t u^{\mathrm{s}}_\varepsilon &= J\,\nabla H(u^{\mathrm{s}}_\varepsilon) + \varepsilon \mathrm{G}_I(u^{\mathrm{s}}_\varepsilon, \mu) \end{aligned} \tag{4.7}$$

Differentiating $\mathcal{D}^{\mathrm{u}}(t)$, we have

$$\begin{aligned} \dot{\mathcal{D}}^{\mathrm{u}}_i(t) &= \varepsilon \langle\, \nabla\Delta(u),\, \mathrm{G}_I(u)\,\rangle + \langle\, \nabla^2\Delta(u)\, J\,\nabla H(u),\, u^{\mathrm{u}}_\varepsilon - u\,\rangle \\ &+ \langle\, \nabla\Delta(u),\, \nabla(J\,\nabla H(u))\,(u^{\mathrm{u}}_\varepsilon - u)\,\rangle + \langle \nabla\Delta(u),\, \mathcal{S}(u^{\mathrm{u}}_\varepsilon, u)\,\rangle \end{aligned}$$

where

$$J\,\nabla H(u^{\mathrm{u}}_\varepsilon) = J\,\nabla H(u) + \nabla(J\,\nabla H(u))\,(u^{\mathrm{u}}_\varepsilon - u) + \mathcal{S}(u^{\mathrm{u}}_\varepsilon, u)$$

we leave out the argument t of the functions for the sake of a less cumbersome notation. From (2.17) we have $\langle J\nabla H, \nabla\Delta\rangle = 0$, differentiating with respect to u gives

$$\nabla(J\nabla H)^{\perp}\,\nabla\Delta + \nabla(\nabla\Delta)\,J\nabla H = 0 \tag{4.8}$$

where $^{\perp}$ denotes the matrix transpose. Taking the inner product of (4.8) with $u^{\mathrm{u}}_\varepsilon - u$ gives

$$\langle \nabla^2\Delta J\,\nabla H, u^{\mathrm{u}}_\varepsilon - u\rangle + \langle \nabla\Delta,\, \nabla(J\,\nabla H)\,(\,u^{\mathrm{u}}_\varepsilon - u\,)\,\rangle = 0$$

and thus,

$$\dot{\mathcal{D}}^{\mathrm{u}}(t) = \varepsilon\langle\, \nabla\Delta(u),\, \mathrm{G}_I(u)\,\rangle + \langle\, \nabla\Delta(u),\, \mathcal{S}(u^{\mathrm{u}}_\varepsilon, u)\,\rangle \tag{4.9}$$

There exists $t_* > 0$ for $t \in (-\infty, t_*)$ such that

$$\|\nabla\Delta(u(t))\|_{\mathbb{H}^1} < C\mathrm{e}^{\sigma t}, \quad \sigma := \sqrt{1 - (2\pi c/L)^2} \tag{4.10}$$

since the $\nabla\Delta$ is evaluated on the homoclinic solution which satisfies the condition (2.21) and (4.3) entails

$$\|\mathcal{S}\|_{\mathbb{H}^1} \le \|u^{\mathrm{u}} - u\|^2_{\mathbb{H}^1} \le C_1\mathrm{e}^{-2\delta t}\varepsilon^2 \tag{4.11}$$

with $2\delta < \sigma$. Now, we want to find out the limit $\lim_{t\to-\infty}\mathcal{D}^{\mathrm{u}}(t)$ in order to represent $\mathcal{D}(0)$ by

$$\mathcal{D}^{\mathrm{u}}(-\infty) + \int_{-\infty}^{0} \dot{\mathcal{D}}^{\mathrm{u}}(t)\,\mathrm{d}t \tag{4.12}$$

We calculate $\lim_{t\to-\infty}\mathcal{D}^{\mathrm{u}}(t)$ and show that $\lim_{t\to-\infty}\mathcal{D}^{\mathrm{u}}(t) = 0$. We have

$$\lim_{t\to-\infty}\mathcal{D}^{\mathrm{u}}(t) = \lim_{t\to-\infty}\langle \nabla\Delta(u(t)),\, u^{\mathrm{u}}_\varepsilon(t) - u(t)\,\rangle$$

There are C_2, $C_3 > 0$ such that

$$\|u^{\mathrm{u}}_\varepsilon(\tau - t^{\mathrm{u}}_\varepsilon) - \mathrm{F}^{\tau}_\varepsilon(u)\|_{\mathbb{H}^1} \;\le\; C_2\mathrm{e}^{\lambda_1\tau}$$

$$\|u(\tau - t_0^{\rm u}) - {\rm F}_\varepsilon^\tau(u)\|_{\mathbb{H}^1} \;\;\leq\;\; C_3 {\rm e}^{\lambda_1 \tau} \tag{4.13}$$

for all $\tau \in (-\infty, 0]$, where $\lambda_1 > 0$, $u_\varepsilon^{\rm u}(-t_\varepsilon^{\rm u}) = u_\varepsilon^{\rm u}$, $u(-t_0^{\rm u}) = u^{\rm u}$ and

$${\rm F}_\varepsilon^\tau(u) \to O_\varepsilon \quad \text{as} \quad \tau \to -\infty \tag{4.14}$$

$${\rm F}_0^\tau(u) = O \in S \tag{4.15}$$

By these relations (4.13-4.15), we obtain that as $t \to -\infty$

$$\|u_\varepsilon^{\rm u}(t) - u(t)\|_{\mathbb{H}^1} \sim {\rm O}(t) \tag{4.16}$$

From relations (4.10) and (4.16) we see that

$$\lim_{t \to -\infty} \mathcal{D}^{\rm u}(t) = 0$$

All this finally yields upon integration of (4.9)

$$\mathcal{D}^{\rm u}(0) = \varepsilon \int_{-\infty}^0 \langle \nabla\Delta(u(t)),\, {\rm G}_I(u(t))\rangle\, {\rm d}t + {\rm O}(\varepsilon^2) \tag{4.17}$$

Similarly, we have

$$\mathcal{D}^{\rm s}(0) = -\varepsilon \int_0^\infty \langle \nabla\Delta(u(t)),\, {\rm G}_I(u(t))\rangle\, {\rm d}t + {\rm O}(\varepsilon^2) \tag{4.18}$$

whence the distance function (4.5) is given by

$$\mathcal{D} = \mathcal{D}^{\rm u}(0) - \mathcal{D}^{\rm s}(0) = \varepsilon\, M_I + {\rm O}(\varepsilon^2) \tag{4.19}$$

with

$$M_I = \int_{-\infty}^\infty \langle \nabla\Delta(u(t)), {\rm G}_I(u(t))\rangle\, {\rm d}t, \quad i = 1, 2 \tag{4.20}$$

where $u(t) = U(x,t)$ the homoclinic solution of the unperturbed system.

The Mel'nikov integral is convergent. Because, the perturbation term ${\rm G}_I = bg_1(u) - au_t$ is exponential decreasing expression, since we have assumed that $g_1(u)$ satisfies the conditions $g_1(0) = 0, \partial_u g_1(0) = 0$ and in the neighborhood of zero

$$\|g_1(u)\|_{\mathbb{H}^1} \leq \mu_9 \|u\|^2 \leq \mu_9 {\rm e}^{-2\sigma t} \tag{4.21}$$

evaluated on the homoclinic orbit, with $\mu_9 > 0$. Also, hold

$$\|u_t\|_{\mathbb{H}^1} \leq \mu_{10} {\rm e}^{-\sigma t}$$

and (4.10). Thus, obtaining convergent integral M. □

Using the integral of motion, we may derive for the case I, explicit formula for the Mel'nikov function

$$M_I(\,\beta, \mu\,) = \int_{-\infty}^\infty \langle\, \nabla\Delta(U(x,t)), {\rm G}_I(U(x,t))\,\rangle\, {\rm d}t \tag{4.22}$$

evaluated on the homoclinic orbit $U(x,t)$. Thus,

$$M_I(\,\beta,\mu\,) = b\int_{-\infty}^{\infty}\int_0^L \frac{\delta\Delta}{\delta u_t}(U)g_1(U)\,\mathrm{d}x\,\mathrm{d}t - a\int_{-\infty}^{\infty}\int_0^L \frac{\delta\Delta}{\delta u_t}(U)U_t\,\mathrm{d}x\,\mathrm{d}t \tag{4.23}$$

with β defined in (2.17) and

$$\frac{\delta\Delta}{\delta u_t} = -\frac{\mathrm{i}}{4c}\Big(\frac{1}{2\sin 2\beta}\Big)^2\Big[\mathrm{E}\frac{\psi_2}{\psi_1} + \mathrm{H}\frac{\psi_1}{\psi_2}\Big] \tag{4.24}$$

where E,H are smooth functions of the general solutions ψ_1,ψ_2 of the Lax pair (2.7).

Setting $M_I(\,\beta,\mu\,)=0$, we obtain an algebraic equation for parameters

$$b - a\kappa = 0 \tag{4.25}$$

where

$$\kappa(\beta) = \int_{-\infty}^{\infty}\int_0^L \frac{\delta\Delta}{\delta u_t}(U)U_t\,\mathrm{d}x\,\mathrm{d}t\,\bigg(\int_{-\infty}^{\infty}\int_0^L \frac{\delta\Delta}{\delta u_t}(U)g_1(U)\,\mathrm{d}x\,\mathrm{d}t\bigg)^{-1}$$

We denote the surface defined by (4.25) by $\mathcal{E}_\beta$

$$\mathcal{E}_\beta\;:\;\beta_0 = \mathcal{B}(b_0,a_0;c)$$

There exists a region $\mathcal{R}$ of the surface $\mathcal{E}_\beta$ such that

$$\frac{\partial M_I}{\partial\beta_0}(\beta_0,b_0,a_0;c)\neq 0 \tag{4.26}$$

and $\left\|\frac{\partial M_I}{\partial\beta_0}\right\| < l$, for $l>0$. By the implicit function theorem there is a neighborhood $\hat{\mathcal{R}}$ of $(b_0,a_0;c)$ and a unique C^{n-2} function

$$\beta_0 = \mathcal{B}(b_0,a_0;\varepsilon,c)$$

defined in $\hat{\mathcal{R}}$ such that

$$\hat{\mathcal{B}}(b_0,a_0;0,c) = \hat{\beta}_0$$

and $\mathcal{D}(\hat{\mathcal{B}}(b_0,a_0;\varepsilon,c),b_0,a_0;\varepsilon,c)=0$. Since $\mathcal{D}$ (the distance of stable and unstable manifolds) is C^{n-2} smooth function by (4.26), we obtain

$$\frac{\partial}{\partial\beta_0}\mathcal{D}(\hat{\mathcal{B}}(b_0,a_0;\varepsilon,c),b_0,a_0;\varepsilon,c)\neq 0$$

and $\left\|\frac{\partial\mathcal{D}}{\partial\beta_0}\right\| < m, m>0$, for $(b_0,a_0;\varepsilon,c)\in\hat{\mathcal{R}}$. Then, $W^{\mathrm{s}}(\mathcal{M}_\varepsilon)$ and $W^{\mathrm{u}}(\mathcal{M}_\varepsilon)$ intersect transversely in the neighborhood of $\hat{\mathcal{R}}$.

Proposition 4.2 *There exist a neighborhood $\hat{\mathcal{R}}$ for the parameter b,a,c and a positive constant $\varepsilon_0>0$ such that fprany $|\varepsilon|<\varepsilon_0$ and $(b,a,c)\in\hat{\mathcal{R}}$ the $W^{\mathrm{s}}(\mathcal{M}_\varepsilon)$ and $W^{\mathrm{u}}(\mathcal{M}_\varepsilon)$ intersect transversely.*

4.2 Case II: Time-Periodic Perturbation

The system (1.1b) can be rewritten in the more general form as follows:

$$\begin{aligned} \dot{\mathrm{x}} &= \mathrm{f}(\mathrm{x}) + \varepsilon \mathrm{G}_{II}(\mathrm{x}, \theta), \\ \dot{\theta} &= 1 \end{aligned} \tag{4.27}$$

We assume that $\mathrm{G}_{II} \in C^{\infty}(\mathbb{H}^1 \times \mathbb{S}^1, \mathbb{H}^1)$, where $\mathbb{S}^1 = \mathbb{R}/T$. This implies that the associated suspended autonomous system on $\mathbb{H}^1 \times \mathbb{S}^1$ has a smooth local flow $\mathrm{F}_{\varepsilon}^{t} : \mathbb{H}^1 \times \mathbb{S}^1 \to \mathbb{H}^1 \times \mathbb{S}^1$, this means that $\mathrm{F}_{\varepsilon}^{t}$ is a smooth map defined for small $|t|$ which is jointly continuous in all variables $\varepsilon, t \in \mathbb{R}, \mathrm{x} \in \mathbb{H}^1, \theta \in \mathbb{S}^1$.

We recall that for $\varepsilon > 0$ small, there is a unique fixed point p_{ε} of P^{ε} near $p_0 = O = (0, 0)$, moreover $p_{\varepsilon} - p_0 = \mathrm{O}(\varepsilon)$, i.e there is a constant K such that $\|p_{\varepsilon}\| < K\varepsilon$, $\forall \varepsilon > 0$.

We consider a neighborhood of the codimension 2 center manifold $\mathcal{M}_{\varepsilon}$, in which we attempt to establish the transversal intersection of $W^{\mathrm{s}}(p_{\varepsilon}) \subset W^{\mathrm{s}}(\mathcal{M}_{\varepsilon})$ and $W^{\mathrm{u}}(p_{\varepsilon}) \subset W^{\mathrm{u}}(\mathcal{M}_{\varepsilon})$. Based on the assumptions and preliminary results of Lemma 3.1, we proceed to calculate the distance $\mathcal{D}$ between the perturbed manifolds $W^{\mathrm{s}}(p_{\varepsilon}), W^{\mathrm{u}}(p_{\varepsilon})$, by calculating the $\mathrm{O}(\varepsilon)$ components of perturbed solution curves of equation (4.27) from the first variation equation. We expand solution curves in $W^{\mathrm{s}}(q_{\varepsilon})$ and $W^{\mathrm{u}}(q_{\varepsilon})$, as we described in Lemma 3.1, points in $W^{\mathrm{s,u}}(p_{\varepsilon})$ are obtained by intersecting $W^{\mathrm{s,u}}(q_{\varepsilon})$ with the section $\mathbb{H}^1 \times \{t_0\}$.

Therefore, (4.27) possesses the perturbed orbit

$$\mathrm{x}_{\varepsilon}^{\mathrm{u}}(t, t_0) = \mathrm{x}_0^{\mathrm{u}}(t - t_0) + \varepsilon \mathrm{x}_1^{\mathrm{u}}(t, t_0) + \mathrm{O}(\varepsilon^2)$$

on the perturbed unstable manifold, where $\mathrm{x}_1^{\mathrm{u}}(t, t_0)$ is the solution of the first variation equation

$$\dot{\mathrm{x}}_1^{\mathrm{u}}(t, t_0) = D\mathrm{f}(\mathrm{x}_0(t - t_0))\mathrm{x}_1^{\mathrm{u}}(t, t_0) + \mathrm{G}_{II}(\mathrm{x}_0(t - t_0), t)$$

and a similar one lying on the perturbed stable manifold.

As in the case I, the distance function is given by:

$$\mathcal{D}(t_0) = \varepsilon M_{II}(t_0) + \mathrm{O}(\varepsilon^2) \tag{4.28}$$

with

$$M_{II}(t_0) = \int_{-\infty}^{\infty} \int_0^L \frac{\delta \Delta}{\delta u_t}(U)\, (\, g_2(t - t_0) - aU_t)\, \mathrm{d}x\, \mathrm{d}t \tag{4.29}$$

evaluated on the homoclinic orbit U.

For $g_2(t) = \Gamma \mathrm{cos} \omega t$, the equation (4.29) takes the form:

$$M_{II}(t_0) = \int_{-\infty}^{\infty} \int_0^L \frac{\delta \Delta}{\delta u_t}(U)\, (\, \Gamma \mathrm{cos} \omega (t - t_0) - aU_t)\, \mathrm{d}x\, \mathrm{d}t \tag{4.30}$$

The Mel'nikov integral in equation (4.30) is convergent. We therefore consider the improper integral as the following limit of a sequence in time $\{\tau_n^{\mathrm{s,u}}\} = \{\pm 2\pi n/\omega\}$, $n = 1, 2, \ldots$

$$M_{II}(t_0, \Gamma, a, \beta, c) = \lim_{n \to \infty} \int_{-\frac{2\pi n}{\omega}}^{\frac{2\pi n}{\omega}} \int_0^L \frac{\delta \Delta}{\delta u_t}(U)\, (\, \Gamma \mathrm{cos} \omega (t - t_0) - aU_t)\, \mathrm{d}x\, \mathrm{d}t \tag{4.31}$$

Since $\frac{\delta\Delta}{\delta u_t}(U)$ and U_t are rapidly decreasing expressions, we are allowed to extend the integration limits in (4.31) to infinity and thus obtaining convergent integral.

We find that the Mel'nikov function is of the form

$$M_{II}(t_0, \Gamma, a, \beta, c) = \mathcal{A}_2 \Gamma \mathrm{sin}\omega t_0 - \mathcal{I}_2 \tag{4.32}$$

where

$$\begin{aligned} \mathcal{A}_2 \mathrm{sin}\,\omega t_0 &= \int_{-\infty}^{\infty}\int_0^L \frac{\delta\Delta}{\delta u_t}(U(x,t))\mathrm{cos}\omega(t-t_0)\,\mathrm{d}x\,\mathrm{d}t \\ \mathcal{I}_2 &= \int_{-\infty}^{\infty}\int_0^L \frac{\delta\Delta}{\delta u_t}(U(x,t))U_t(x,t)\,\mathrm{d}x\,\mathrm{d}t \end{aligned}$$

Setting $M_{II}(t_0) = 0$, we obtain

$$t_0^{\pm} = \pm\frac{1}{\omega}\mathrm{sin}^{-1}(\frac{a}{\Gamma}\chi_2) \quad \chi_2 = \frac{\mathcal{I}_2}{\mathcal{A}_2} \tag{4.33}$$

Lemma 4.1 *For $\frac{1}{4} < c^2 < 1$, there exists an open interval $I \subset \mathbb{R}$, such that for any $\Gamma, a, \beta \in I$, there are two values of t_0: $t_0^{\pm} = t_0^{\pm}(c, \omega, \Gamma, a, \beta, \chi_2)$ (cf. (4.33)) at which $M_{II}(t_0, \mu) = 0$, $\mu = (c, \omega, \Gamma, a, \beta)$. Moreover, when $M_{II}(t_0)$ is viewed as function of $t_0^{\pm}$, these zeros are simple.*

By this lemma, eq. (4.28) and the implicit function theorem, we have the following proposition:

Proposition 4.3 *There exists a subregion $\hat{\mathcal{R}}_1$ of external parameter space*

$$\hat{\mathcal{R}} = \Big\{ (c, \omega, \Gamma, a, \beta) : \frac{1}{4} < c^2 < 1,\, \omega \in (\,0, 0.87\,),\, \Gamma, a \in (\,0.1, 1\,),\, \beta \in (\,-\pi/2, \pi/2\,) \Big\}$$

and a positive number ε_0, such that for any fixed parameters $\{c, \omega, \Gamma, a, \beta\} \in \hat{\mathcal{R}}_1 \times (\,0, \varepsilon_0\,)$, there are $t_0^{\pm}$ (cf. (4.33)) at which $W^{\mathrm{s}}(p_\varepsilon), W^{\mathrm{u}}(p_\varepsilon)$ intersect transversely.

5 Conclusions

In this paper, we have used a Mel'nikov type analysis to detect the splitting of homoclinic manifolds in a perturbed sine–Gordon equation with even spatial periodic boundary conditions. We extended the two-dimensional phase portrait for the pendulum into the infinite dimensional phase space $\mathbb{H}^1$ of the integrable periodic sine–Gordon equation.

This analysis consists first of a study of homoclinic and chaotic dynamics on perturbed SG systems. The prerequisite for such study is that the unperturbed soliton system, viewed as a Hamiltonian system should have a homoclinic structure. In the process of proving the existence of invariant manifolds for such a system several mathematical tools from dynamical systems theory are utilised. We established, via Mel'nikov theory

the existence of transversal intersection of a codimension 2 center-stable and center-unstable manifold, of the perturbed singular point O_ε, which implies the occurrence of chaotic dynamics. In particular geometric settings, a simple zero of the Mel'nikov function, with respect to one of its parameters insure the intersection of stable and unstable manifolds of a critical torus and the persistence which follows as a consequence of this intersection.

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